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Lyapunov exponents and bifurcation current for polynomial-like maps

Ngoc-Mai Pham

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Abstract

We study holomorphic families of polynomial-like maps depending on a parameter s. We prove that the partial sums of largest Lyapunov exponents are plurisubharmonic functions of s. We also study their continuity and introduce the bifurcation locus as the support of bifurcation currents.

1 Introduction

In this paper, we study the dependence of Lyapunov exponents on parameters for polynomial-like maps in any dimension.

Recall that a *polynomial-like map* is a proper holomorphic map $f : U \to V$ where U, V are open subsets of $\mathbb{C}^k$ and $U \Subset V$. In particular, f defines a ramified covering over V. The degree d_t of the covering is the *topological degree* of f, it is equal to $\sharp f^{-1}(z), z \in V$, counting multiplicities. The family of polynomial-like maps is very large. One checks easily that small perturbations of f define also polynomial-like maps.

In [DS1], Dinh-Sibony constructed for such a map the *equilibrium measure* μ as follows: if Ω is an arbitrary smooth probability measure on V then μ is the limit of $d_t^{-n}(f^n)^*\Omega$ in the sense of measures. The measure μ does not depend on the choice of Ω. It is totally invariant: $d_t^{-1}f^*\mu = f_*\mu = \mu$, of maximal entropy and is mixing. Following Oseledec [Os], f admits k Lyapunov exponents with respect to μ that we denote by

$$\chi_1 \geq \chi_2 \geq \cdots \geq \chi_k.$$

They satisfy the following inequality (see [DS1])

$$\chi_1 + \chi_2 + \cdots + \chi_k \geq \frac{1}{2}\log d_t.$$

The support J of μ is called the *Julia set* (of maximal order) of f. It is contained in the boundary of the *filled Julia set*

$$K := \{z \in U : \; f^n(z) \in U \quad \text{for every } n \geq 0\} = \bigcap_{n \geq 1} f^{-n}(V).$$

We recall that the measure μ *maximizes the plurisubharmonic (p.s.h. for short) moments* [DS1, Prop.3.2.6]. That is, if ν is a totally invariant probability measure and φ is a p.s.h. function on U then $\int \varphi \mathrm{d}\nu \leq \int \varphi \mathrm{d}\mu$.

Note that the study of holomorphic endomorphisms of $\mathbb{C}P^k$ of degree algebraic $d \geq 2$ can be reduced to the study of some polynomial-like maps. Indeed, we can lift these maps to $\mathbb{C}^{k+1}$ and the restrictions of the lifted maps to a large ball $V \subset \mathbb{C}^{k+1}$ are polynomial-like maps.

Now, consider a holomorphic family of polynomial-like maps:

$$f_s : U_s \to V_s, s \in \Lambda.$$

More precisely, we have a holomorphic map $F : \mathcal{U} \to \mathcal{V}$, where $\mathcal{U} \subset \mathcal{V}$ are open sets in $\Lambda \times \mathbb{C}^k$ and Λ is a connected complex manifold of dimension m. Define

$$U_s = \mathcal{U} \cap (\{s\} \times \mathbb{C}^k), \quad V_s = \mathcal{V} \cap (\{s\} \times \mathbb{C}^k).$$

We assume that $U_s \Subset V_s$ and that the restriction of F to U_s defines a polynomial-like map $f_s : U_s \to V_s$. We often identify U_s and V_s to open sets in $\mathbb{C}^k$. Observe that the topological degree d_t of f_s does not depend on s.

Let us denote the equilibrium measure of f_s by μ_s, the Julia set by J_s and the filled Julia set by K_s. We order the Lyapunov exponents by

$$\chi_1(s) \geq \chi_2(s) \geq \cdots \geq \chi_k(s).$$

In this paper, we prove that

$$L_p(s) := \chi_1(s) + \chi_2(s) + \cdots + \chi_p(s)$$

is p.s.h. on s for $1 \leq p \leq k$. The case where $p = k$ was proved in [DS1].

We define the *bifurcation current* associated to $(f_s)_{s\in\Lambda}$ by

$$B_F := \mathrm{dd^c} L_k.$$

This is a positive closed $(1,1)$-current on Λ. The support of B_F is called the *bifurcation locus* of $(f_s)_{s\in\Lambda}$. Since L_k is locally bounded ($L_k \geq \frac{1}{2}\log d_t$), we can consider the *higher degree bifurcation currents*:

$$B_F^i := B_F \wedge \ldots \wedge B_F \quad (i \text{ times})$$

and the *higher degree bifurcation locus* $\mathrm{supp}(B_F^i)$ for $0 \leq i \leq m$ (see Definition 2.4). We also introduce in Section 2 the *total bifurcation current* $\hat{B}_F$ on $\mathcal{U}$ such that $\pi_*(\hat{B}_F) = B_F$ where $\pi : \mathcal{U} \to \Lambda$ is the canonical projection (see Definition 2.8).

When the measure μ_{s_0} of f_{s_0} is *PLB* (i.e the p.s.h. functions on V_{s_0} are μ_{s_0}-integrable) we show that L_k is continuous in a neighborhood of s_0. Note that the property "μ_s PLB" is equivalent to the fact that some natural dynamical degree d_s^* of f_s satisfies $d_s^* < d_t$.

We then study the stability of $(f_s)_{s\in\Lambda}$ in Section 4. In the case of dimension 1, we are able to extend the results of Mañé-Sad-Sullivan (see [MSS]) to the case of polynomial-like maps. In particular, we will prove that the stability of J_s, i.e the continuity of J_s in the Hausdorff sense, implies that the unique Lyapunov exponent $\chi(s)$ defines a pluriharmonic function on Λ (see also [DM] for the case of rational maps on $\mathbb{C}P^1$). In the case of higher dimension, we show that if the critical set $\mathcal{C}_s$ of f_s does not intersect J_s and if μ_s is PLB for $s \in \Lambda$ then L_k is pluriharmonic and $(f_s)_{s\in\Lambda}$ is stable. In this case, the bifurcation locus is empty. The condition on $\mathcal{C}_s$ is often easy to check.

Note that in a recent work [BB], Bassanelli-Berteloot gave another sufficient condition for L_k to be pluriharmonic, for holomorphic maps in $\mathbb{C}P^k$ (see also Remark 2.9). A similar problem for Hénon maps was studied by Bedford-Lyubich-Smillie (see [BS], [BLS]).

Observe that if $f : U \to V$ is a polynomial-like map then $f : f^{-1}(V') \to V'$ is also polynomial-like for $U \subset V' \subset V$. The problems on families of maps that we consider are semi-local. Then we can assume to simplify the notation that $V_s = V$ for every s and $\mathcal{V} = \Lambda \times V$ in Section 2 and Section 3.

The readers who are not familiar with the horizontal currents and slicing theory, may consult our Appendix A or [DS2].

2 Partial sums of Lyapunov exponents

In this section, we study the partial sums of largest Lyapunov exponents of polynomial-like maps. We first prove the following useful result.

Proposition 2.1 *Let $(f_s)_{s\in\Lambda}$ be a holomorphic family of polynomial-like maps as above. Then there exists a horizontal positive closed current $\mathcal{R}$ on $\Lambda \times V$ of bidimension (m,m) such that for every $s \in \Lambda$ the slice $\langle \mathcal{R}, \pi, s\rangle$ is equal to μ_s where $\pi : \Lambda \times V \to \Lambda$ is the canonical projection*

Proof. Let S be an arbitrary horizontal current of bidimension (m,m) on $\Lambda \times V$, define $S_s := \langle S, \pi, s\rangle$ the slice of S. Let ϑ be a smooth probability measure with compact support in V. Consider the horizontal positive closed current $S := \pi_V^*(\vartheta)$ of bidimension (m,m) on $\Lambda\times V$, where π_V is the canonical projection of $\Lambda \times V$ on V. Define

$$S_n := \frac{1}{d_t^n}(F^n)^* S.$$

We identify S_s with ϑ. Then $S_{n,s} = \frac{1}{d_t^n}(f_s^n)^*\vartheta$ converges weakly to μ_s for $s \in \Lambda$ (see [DS1]).

Since the masses of S_n are locally uniformly bounded there exists a subsequence $(i_n)_{n\in\mathbb{N}}$ such that S_{i_n} converge to a horizontal positive closed current $\mathcal{R}$ on $\Lambda \times V$. By definition $\mathrm{supp}(\mathcal{R}) \subset \bigcup_{s\in\Lambda} K_s$, where K_s is the filled Julia set of f_s. Let s_0 be a fixed point in Λ. Let U_0 be a subset of U_{s_0} and Λ_0 be a small neighborhood of s_0 such that $\bigcup_{s\in\Lambda_0} K_s \Subset U_0 \Subset \bigcap_{s\in\Lambda_0} U_s$. Consider a smooth p.s.h. function ψ on a neighborhood $\Lambda_0 \times V_0$ and a continuous form Ω of maximal degree with compact support in Λ_0. By formula (8) in the Appendix A, for every n, we have

$$\int_\Lambda S_{i_n,s}(\psi)\Omega(s) = \langle S_{i_n} \wedge \pi^*(\Omega), \psi\rangle.$$

Then

$$\int_\Lambda \mu_s(\psi)\Omega(s) = \langle \mathcal{R} \wedge \pi^*(\Omega), \psi\rangle = \int_\Lambda \mathcal{R}_s(\psi)\Omega(s). \tag{1}$$

Define $u(s) := \mathcal{R}_s(\psi)$ for $s \in \Lambda_0$. By Proposition A.1, u is p.s.h. on Λ_0.

We want to prove that $\mu_s(\psi)$ is also a p.s.h. function. Consider a sequence (s_m) converging to s_0 such that $\mu_{s_m} \rightharpoonup \nu$. Hence $f_{s_0}^* \nu = d_t \nu$. Since ψ is uniformly continuous on $\Lambda_0 \times V_0$ then for $\epsilon > 0$ and s close to s_0:

$$\int \psi(s,z)\mathrm{d}\mu_s \leq \int (\psi(s_0,z)+\epsilon)\mathrm{d}\mu_s.$$

Then

$$\begin{aligned}\limsup_{m\to\infty} \int \psi(s,z)\mathrm{d}\mu_{s_m} &\leq \limsup_{m\to\infty} \int (\psi(s_0,z)+\epsilon)\mathrm{d}\mu_{s_m} \\ &= \int (\psi(s_0,z)+\epsilon)\mathrm{d}\nu.\end{aligned}$$

Since the equilibrium measure μ_{s_0} maximizes p.s.h. moments and ϵ is arbitrarily small, we get:

$$\limsup_{s\to s_0} \mu_s(\psi) \leq \int \psi(s_0,z)\mathrm{d}\nu \leq \int \psi(s_0,z)\mathrm{d}\mu_{s_0} = \mu_{s_0}(\psi).$$

Therefore, $\mu_s(\psi)$ is upper semi-continuous.

Define

$$\begin{aligned}\mu_s^N(\psi) :&= \int \frac{1}{d_t^N}\psi (f_s^N)^*\vartheta \\ &= \int \frac{1}{d_t^N}(f_s^N)_*(\psi)\vartheta.\end{aligned}$$

Recall here that $(f_s^N)_*(\psi)(z) := \sum_{f_s^N(w)=z}\psi(w)$ where the roots of the equation $f_s^N(w) = z$ are counted with multiplicities. Since $\psi(s,z)$ is p.s.h., $(f_s^N)_*(\psi)(s,z)$ is a p.s.h. function of (s,z). Hence $\mu_s^N(\psi)$ is p.s.h. on Λ_0. Since μ_s is the limit of $(f_s^N)^*\vartheta$, as $N \to \infty$, then $\mu_s^N(\psi)$ converges to $\mu_s(\psi)$, as $N \to \infty$. Consequently, $\mu_s(\psi)$ is a p.s.h. function.

The equality (1) is valid for all continuous form Ω of maximal degree with compact support in Λ_0 then $\mu_s(\psi) = u(s)$ a.e on Λ_0. But these functions are p.s.h. hence $\mu_s(\psi) = u(s) = \mathcal{R}_s(\psi)$ for all $s \in \Lambda_0$. We deduce that $\mu_s(\psi) = \mathcal{R}_s(\psi)$ for ψ smooth with compact support in $\Lambda_0 \times V_0$. Indeed, we can write $\psi = \psi_1 - \psi_2$ where ψ_1 and ψ_2 are p.s.h. on $\Lambda_0 \times V_0$. Therefore, $\mathcal{R}_s = \mu_s$ for every $s \in \Lambda$.

$\square$

The following theorem generalizes a result of [DS1].

Theorem 2.2 *Let $(f_s)_{s\in\Lambda}$ be a holomorphic family of polynomial-like maps with topological degree $d_t \geq 2$ as above. Then the function $L_p(s) = \chi_1(s) + \cdots + \chi_p(s)$ is p.s.h. on Λ for all $1 \leq p \leq k$. In particular, L_p is upper semi-continuous.*

Let $f : U \to V$ be a polynomial-like map, $U \Subset V \Subset \mathbb{C}^k$. Let μ denote its equilibrium measure, J its Julia set and χ_i the Lyapunov exponents with $\chi_1 \geq \cdots \geq \chi_k$. Let $Df^n(z)$ denote the differential of f^n at z. This is a linear map from the tangent space T_z of $\mathbb{C}^k$ at z to the one at $f^n(z)$.

Consider an orthonormal family of vectors $\{e_1(z), \ldots, e_p(z)\}$ in T_z and the linear space $e^p(z)$ generated by $e_1(z), \ldots, e_p(z)$ for $z \in f^{-n}(U)$. The volume $\lambda(e^p(z), Df^n)$ of the parallelotope determined by the vectors

$$Df^n(e_1(z)), \ldots, Df^n(e_p(z))$$

is called the *coefficient of expansion* in the direction $e^p(z)$. It depends only on $e^p(z)$, not on the choice generators. If $e^{p+q}(z) = e^p(z) \oplus e^q(z)$, we have

$$\lambda(e^{p+q}(z), Df^n) \leq \lambda(e^p(z), Df^n)\, \lambda(e^q(z), Df^n).$$

A p-vector v in the exterior product space $\bigwedge^p T_z$ is *simple* if it can be written as $v = e_1 \wedge \ldots \wedge e_p$ where $e_1, \ldots, e_p$ are vectors in T_z. Simple p-vectors generate $\bigwedge^p T_z$. The map f induces a linear map

$$\wedge^p Df^n(z) : \bigwedge^p T_z \to \bigwedge^p T_{f^n(z)}$$

which is defined by

$$\wedge^p Df^n(z)(v) := Df^n(e_1) \wedge \ldots \wedge Df^n(e_p).$$

For $z \in J$ define the Lyapunov p-dimensional characteristic number of $e^p(z)$ by:

$$\chi(e^p(z)) := \limsup_{n\to\infty} \frac{1}{n} \log \lambda(e^p(z), Df^n).$$

By [Os], there exists a regular subset E of J satisfying $f(E) \subset E$, $\mu(E) = 1$ such that for all $z \in E$, $1 \leq p \leq k$

$$\chi(e^p(z)) = \lim_{n\to\infty} \frac{1}{n} \log \lambda(e^p(z), Df^n)$$

$$\chi_1 + \cdots + \chi_p = \sup_{e^p(z)} \chi(e^p(z)).$$

Define $\|\wedge^p Df^n(z)\| := \sup_{e^p(z)} \lambda(e^p(z), Df^n)$. We deduce from the previous discussion that for $1 \leq p \leq k$ and $z \in E$,

$$\chi_1 + \cdots + \chi_p = \lim_{n\to\infty} \frac{1}{n} \log \|\wedge^p Df^n(z)\| \tag{2}$$

and

$$\left\|\wedge^p Df^{m+n}(z)\right\| \leq \|\wedge^p Df^n(z)\| \, \|\wedge^p Df^m(f^n(z))\| \, . \tag{3}$$

Proof of Theorem 2.2. Define $\psi_{p,n}(s,z) := \frac{1}{n}\log \|\bigwedge^p Df_s^n(z)\|$ and $\varphi_{p,n}(s) := \int \psi_{p,n}(s,z)\mathrm{d}\mu_s$ for all $z \in f_s^{-n}U_s$. It is clear that $\psi_{p,n}(s,z)$ is p.s.h. on a sufficiently small neighborhood of $\bigcup_{s\in\Lambda} K_s$. By formula (2), we have

$$L_p(s) = \lim_{n\to\infty} \varphi_{p,n}(s).$$

Proposition 2.1 and A.1 imply that $\varphi_{p,n}$ is p.s.h. on Λ. The inequality (3) implies that $\{\varphi_{p,2^n}\}$ is a decreasing sequence of p.s.h. functions. Then the limit $L_p(s)$ is also p.s.h. on Λ. $\square$

Remark 2.3 Let $\mathcal{I}$ be a subset of $\{1, 2, \ldots, k\}$ and $L_{\mathcal{I}}(s) := \sum_{i\in\mathcal{I}} \chi_i(s)$. If $L_{\mathcal{I}}$ is a p.s.h. function of s for any holomorphic family (f_s) of polynomial-like maps then $L_{\mathcal{I}}$ is one of the previous sums $L_1, L_2, \ldots, L_k$. To prove this, we can use the following family of maps $f_s : \mathbb{C}^k \to \mathbb{C}^k$ by

$$(z_1, z_2, \ldots, z_k) \mapsto (f_{1,s}(z_1), f_{2,s}(z_2), \ldots, f_{k,s}(z_k)),$$

where $(f_{i,s})$ is a holomorphic family of polynomial maps in dimension 1 for $1 \leq i \leq k$.

Definition 2.4 Theorem 2.2 allows us to define the following positive closed current associated to $(f_s)_{s\in\Lambda}$

$$B_F := \mathrm{dd^c} L_k.$$

We call it the *bifurcation current* and its support the *bifurcation locus*. Since $L_k \geq \frac{1}{2} d_t$, we can define the *higher degree bifurcation current*

$$B_F^i := B_F \wedge \ldots \wedge B_F \quad (i \quad \text{times})$$

and the *higher degree bifurcation locus* as support of B_F^i for $1 \leq i \leq m$.

Remark 2.5 In [DM], DeMarco considered holomorphic families of rational maps on $\mathbb{C}P^1$. She proved that the bifurcation locus and the complement of the *stable set* coincide. The stable set is the largest open subset of Λ where the Julia set depends continuously on the parameter.

We also have the following stronger "variation" of Theorem 2.2.

Corollary 2.6 *Let $(f_s)_{s\in\Lambda}$ be a holomorphic family of polynomial-like maps as above. Then*

$$L_{p,\lambda}(s) := \max(\chi_1(s), \lambda) + \max(\chi_2(s), \lambda) + \cdots + \max(\chi_p(s), \lambda)$$

is a p.s.h. function on Λ for all $\lambda \in \mathbb{R} \cup \{-\infty\}$.

Proof. It is clear that

$$L_{p,\lambda}(s) = \max\{L_p(s), L_{p-1}(s) + \lambda, \ldots, L_1(s) + (p-1)\lambda, p\lambda\}.$$

The corollary follows.

□

We can chose S in Proposition 2.1 such that $\mathcal{R}$ has support in $\partial\mathcal{K}$, where $\mathcal{K} := \bigcup_{s\in\Lambda} K_s$. Note that K_s depends upper semi-continuously on s and $\mathcal{K}$ is closed. Using Cesàro means, we can construct a positive closed current R such that $F^*(R) = d_t R$ and $R_s = \mu_s$. More precisely, R is a limit of a subsequence of $(\mathcal{S}_n)$, where $\mathcal{S}_n := \frac{1}{n}\sum_{i=1}^n S_n$. A horizontal positive closed current $\mathcal{R}$ such that $\text{supp}(\mathcal{R}) \subset \partial\mathcal{K}$, $F^*(\mathcal{R}) = d_t\mathcal{R}$ and $\mathcal{R}_s = \mu_s$ is called *equilibrium current* associated to the family $(f_s)_{s\in\Lambda}$.

Remark 2.7 In the case $\dim V = 1$, the equilibrium current $\mathcal{R}$ is unique. More precisely, the function $u(s,w) = \int_{\{s\}\times V} \log|w-z| \mathrm{d}\mu_s(z)$ is a p.s.h. potential for $\mathcal{R}$. Let S be a horizontal positive closed current with slice mass unit such that $\text{supp}(S_s)$ is not a polar set in V for all $s \in \Lambda$ then $\frac{1}{d_t^n}(F^n)^*S$ converges weakly to $\mathcal{R}$.

Define $J(s,z) := \det \text{Jac} f_s(z)$ then $\mathrm{dd^c}(\log|J|) = [\mathcal{C}_F]$, where $\mathcal{C}_F$ is the critical set of F. Let $\mathcal{R}$ be an equilibrium current of (f_s) on $\Lambda \times V$. Note that $\mathcal{R}_s(\log|J|) = \mu_s(\log|J|) = L_k \geq \frac{1}{2}d_t$. Theorem A.2 in the Appendix A implies that the current $\log|J|\mathcal{R}$ is well defined.

Definition 2.8 We define the *total bifurcation current* associated to $(f_s)_{s\in\Lambda}$ as

$$\hat{B}_F := \mathrm{dd^c}(\log|J|\mathcal{R}) = [\mathcal{C}_F] \wedge \mathcal{R}$$

and *total bifurcation locus* as its support. Then the current $\hat{B}_F$ is positive closed. Observe that $B_F = \pi_*(\hat{B}_F)$. As an immediate consequence, if $\partial\mathcal{K} \cap [\mathcal{C}_F] = \varnothing$ then the bifurcation locus, total bifurcation locus are empty and L_k is p.s.h. (see also Theorem 4.3).

Remark 2.9 If $F = (f_s)_{s\in\Lambda}$ is a holomorphic family of holomorphic endomorphisms in $\mathbb{C}P^k$, we can find a horizontal $(1,1)$-current τ on $\Lambda \times \mathbb{C}P^k$ such that the slices $\langle \tau, \pi, s\rangle$ are the Green currents associated to f_s. Moreover τ has local continuous potentials. Then we can define *total j-bifurcation currents* by

$$\hat{B}_F^{(j)} := [\mathcal{C}_F] \wedge \tau^j \quad \text{and} \quad B_F^{(j)} := \pi_*(\hat{B}_F^{(j)}), \quad 1 \leq j \leq k.$$

The currents $\hat{B}_F^{(k)}$ and $B_F^{(k)}$ correspond to the currents $\hat{B}_F$ and B_F defined previously. These definitions are the starting point of our study which was developed independently by Bassanelli-Berteloot (see [BB]). Note that Bassanelli-Berteloot also obtained a nice formula for the sum of all the Lyapunov exponents of endomorphisms of $\mathbb{C}P^k$ which generalizes formula obtained by DeMarco (see [DM]) in the one variable case.

3 Continuity of the sum of the exponents

In this section we give a necessary condition so that the sum of all the Lyapunov exponents depends continuously on the parameter. We use here a notion of measure PLB introduced by Dinh-Sibony in [DS1].

Recall that a positive measure ν with compact support in an open set $U \subset \mathbb{C}^k$ is said to be *PLB* in U if p.s.h. functions on U are ν-integrable. In other words, if φ is p.s.h. on U we have: $\int \varphi \mathrm{d}\nu > -\infty$. The following theorem is stronger than the fact that $\mu_s \rightharpoonup \mu_{s_0}$.

Theorem 3.1 *Let f_s, μ_s be as above. Assume that V is Stein and μ_{s_0} is PLB. Then $\mu_s(\varphi) \to \mu_{s_0}(\varphi)$, as $s \to s_0$ for all p.s.h. function φ on V.*

Note that in [DS1], the authors proved that "μ_s PLB" is stable under small perturbations. In the case of dimension 1, the equilibrium measure of

any polynomial-like map is PLB. In the context of Theorem 3.1, μ_s is PLB if s close enough to s_0.

Since the problem here is local for s, we can replace V by a small perturbation such that the boundary ∂V is real analytic. Then the boundaries of U_s and $\mathcal{U}$ are also real analytic for $s \in \Lambda_0$. Choose a neighborhood Λ_0 of s_0 in Λ and a Stein open set W with smooth boundary such that:

$$\bigcup_{s\in\Lambda_0} f_s^{-2}(V) \Subset W \Subset \bigcap_{s\in\Lambda_0} U_s.$$

The following proposition and lemmas 3.1- 3.4 are refinements of the results in [DS1]. We refer to that paper for the proofs.

Proposition 3.2 *Let p be a positive integer. Then*

(i) For $s \in \Lambda_0$ the norm of the operator

$$\mathcal{L}_s := d_t^{-1}(f_s)_* : \mathrm{PSH}(W) \cap \mathrm{L}^2(W) \to \mathrm{PSH}(U_s) \cap \mathrm{L}^2(U_s)$$

is uniformly bounded by a constant A.

(ii) There exist a positive integer $n_0 \geq 2$, a constant $0 < c < 1$ and a neighborhood Λ_1 of s_0, $\Lambda_1 \subset \Lambda_0$ such that if φ is a p.s.h. function on V then

$$\|\mathcal{L}_s^{nn_0}\mathrm{dd^c}\varphi\|_{U_{s_0}} \leq c^n \|\mathrm{dd^c}\varphi\|_{U_{s_0}},$$

for $s \in \Lambda_1$.

To prove Theorem 3.1, we can replace f_s by $f_s^{n_0}$. Then there exists $\tilde{U}$ so that for $s \in \Lambda_1$, we have

$$\bigcup_{s\in\Lambda_1} U_s \Subset \tilde{U} \Subset V;\ \|\mathcal{L}_s^{n}\mathrm{dd^c}\varphi\|_{\tilde{U}} \leq c^n \|\mathrm{dd^c}\varphi\|_{\tilde{U}}. \tag{4}$$

Proposition 3.3 *Let $U \Subset V$ be two open sets in $\mathbb{C}^k$. Let $(\nu_\theta)_{\theta\in\Gamma}$ be a family of probability measures supported in a compact set $K \subset U$. Suppose that there is a constant $B > 0$ such that for all p.s.h. function ψ on U and for all $\theta \in \Gamma$, we have: $\|\psi\|_{\mathrm{L}^1(\nu_\theta)} \leq B\|\psi\|_{\mathrm{L}^2(U)}$. Then there exists a constant $0 < b < 1$ such that*

$$\sup_U \psi \leq b \sup_V \psi$$

for all p.s.h. function ψ on V satisfying: $\int \psi \mathrm{d}\nu_\theta = 0$ for at least one θ.

Let H denote the subspace of pluriharmonic functions in $\mathrm{L}^2(W)$ and $H^\perp$ the cone of p.s.h. functions orthogonal to H. Let φ be a p.s.h. function on W and $\varphi = u+v$ with $u \in H$ and $v \in H^\perp$ the canonical decomposition of φ. We also have $\mathcal{L}_s\varphi = \mathcal{L}_{1,s}u + \mathcal{L}_{2,s}v + \mathcal{L}_{3,s}v$, where $\mathcal{L}_{1,s} : H \to H$, $\mathcal{L}_{2,s} : H^\perp \to H$ and $\mathcal{L}_{3,s} : H^\perp \to H^\perp$ are canonical linear maps associated to $\mathcal{L}_s$.
Following Proposition 3.2, for $s \in \Lambda_0$, we have

$$\|\mathcal{L}_{2,s}\|, \|\mathcal{L}_{3,s}\| \leq A \quad \text{for} \quad s \in \Lambda_0. \tag{5}$$

We have

$$\mathcal{L}_s^n\varphi = \mathcal{L}_{1,s}^n u + \mathcal{L}_{1,s}^{n-1}\mathcal{L}_{2,s}v + \mathcal{L}_{1,s}^{n-2}\mathcal{L}_{2,s}\mathcal{L}_{3,s}v + \cdots + \mathcal{L}_{2,s}\mathcal{L}_{3,s}^{n-1}v + \mathcal{L}_{3,s}^n v.$$

The following lemma is a consequence of the solution of $\partial\bar\partial$-equation in a Stein open set.

Lemma 3.4 *There exists a constant $C > 0$ such that for $s \in \Lambda_0$, $v \in H^\perp$, we have*

$$\|\mathcal{L}_{3,s}v\|_{\mathrm{L}^2(W)} \leq C \|\mathrm{dd^c} v\|_W .$$

We have an easy following lemma.

Lemma 3.5 *Let $K \Subset U \subset \mathbb{C}^k$ then there exists a positive constant $A(K,U)$ such that for all φ p.s.h. on U, we have*

$$\|\varphi\|_{\mathrm{L}^2(U)} \geq A(K,U) \|\mathrm{dd^c}\varphi\|_K .$$

Proof. Let Φ be a positive form with compact support in U which is equal to $(\mathrm{dd^c}\|z\|^2)^{k-1}$ on a neighborhood of K. Then we have $\|\mathrm{dd^c}\varphi\|_K \leq \langle \mathrm{dd^c}\varphi, \Phi\rangle = \int_U \varphi \mathrm{dd^c}\Phi$. It is clear that there exists a positive constant $A(K,U)$ such that $\int_U \varphi\mathrm{dd^c}\Phi \leq A(K,U)\|\varphi\|_{\mathrm{L}^2(U)}$. Lemma 3.5 follows.

$\square$

Lemma 3.6 *Let U_0 be an open subset of V satisfying $\tilde U \Subset U_0 \Subset V$. Then there exists a constant $A_1 > 0$ such that for $s \in \Lambda_1$ and φ p.s.h. on V, we have*

$$\|\varphi\|_{\mathrm{L}^1(\mu_s)} \leq A_1 \|\varphi\|_{\mathrm{L}^2(U_0)} .$$

Proof. Define $b(s) := \int u \mathrm{d}\mu_s$, $b_j(s) := \int \mathcal{L}_{2,s}\mathcal{L}_{3,s}^j v \mathrm{d}\mu_s$ and

$$h_n(s) := b(s) + b_0(s) + \cdots + b_{n-1}(s).$$

It is prove that in [DS1] that the sequence $(h_n(s))$ converge to $\mu_s(\varphi)$.

Inequality (4) and Lemma 3.4 imply

$$\begin{aligned} \left\|\mathcal{L}_{3,s}^j v\right\|_{\mathrm{L}^2(W)} &\leq C\left\|\mathrm{dd^c}\mathcal{L}_{3,s}^{j-1}v\right\|_W \\ &= C\left\|\mathrm{dd^c}\mathcal{L}_s^{j-1}\varphi\right\|_W \\ &\leq A_2 c^j \left\|\mathrm{dd^c}\varphi\right\|_{\tilde{U}}, \end{aligned}$$

where $A_2 = C/c$ does not depend on s and φ.

Since $\mathcal{L}_{2,s}\mathcal{L}_{3,s}^j v$ is pluriharmonic (for each s fixed), we deduce from the last inequality and the inequality (5) that

$$|b_j(s)| \leq A_3 c^j \left\|\mathrm{dd^c}\varphi\right\|_{\tilde{U}} \quad \text{and} \quad |b(s)| \leq A_3 \left\|\varphi\right\|_{\mathrm{L}^2(\tilde{U})},$$

where A_3 is independent of s and φ. The second inequality is a consequence of the pluriharmonicity of u. By Lemma 3.5 and the inequalities above, there exists a constant $A_4 > 0$ such that

$$\mu_s(\varphi) \geq -A_4 \left\|\varphi\right\|_{\mathrm{L}^2(U_0)}. \tag{6}$$

Define $\varphi^+ := \max(\varphi, 0)$ then

$$\int |\varphi| \mathrm{d}\mu_s = \int (-\varphi + 2\varphi^+)\mathrm{d}\mu_s \leq \left|\mu_s(\varphi)\right| + 2\sup_W \varphi^+.$$

The submean inequality implies that $\mu_s(\varphi) \leq \sup_W \varphi^+ \leq A_5 \left\|\varphi\right\|_{\mathrm{L}^2(U_0)}$, where A_5 is independent of s and φ. By inequality (6), we have

$$\left\|\varphi\right\|_{\mathrm{L}^1(\mu_s)} \leq A_1 \left\|\varphi\right\|_{\mathrm{L}^2(U_0)},$$

where $A_1 := \max(A_4, 3A_5)$ is independent of s and φ.

□

Proof of Theorem 3.1. Let V_0 be an open subset of V such that $U_0 \Subset V_0 \Subset V$. By Lemma 3.6, the family $(\mu_s)_{s\in\Lambda_1}$ satisfies the hypothesis of Proposition 3.3. Then there exists $0 < c_0 < 1$, (independent of $s \in \Lambda_1$ and φ), such that

$$\sup_{V_0}(\mathcal{L}_s^{n+1}\varphi - \mu_s(\varphi)) \leq c_0 \sup_{V_0}(\mathcal{L}_s^n\varphi - \mu_s(\varphi)).$$

Hence

$$\sup_{V_0}\mathcal{L}_s^n\varphi - \mu_s(\varphi) \leq c_0^{n-1}(\sup_{U_0}\varphi - \mu_s(\varphi)).$$

By Lemma 3.6, there exists a constant $M > 0$ (independent of $s \in \Lambda_1$ and φ) such that

$$\sup_{V_0}\mathcal{L}_s^n\varphi - \mu_s(\varphi) \leq M\,\|\varphi\|_{\mathrm{L}^2(V_0)}\,c_0^n.$$

Since $W \Subset V_0$ and $\mu_s(\mathcal{L}_s^n\varphi) = \mu_s(\varphi)$, we obtain

$$0 \leq \sup_{W}\mathcal{L}_s^n\varphi - \mu_s(\varphi) \leq M\,\|\varphi\|_{\mathrm{L}^2(V_0)}\,c_0^n. \tag{7}$$

Then

$$|\mu_s(\varphi) - \mu_{s_0}(\varphi)| \leq |\sup_W \mathcal{L}_s^n\varphi - \sup_W \mathcal{L}_{s_0}^n\varphi| + M\,\|\varphi\|_{\mathrm{L}^2(V_0)}\,c_0^n.$$

Define $\mathcal{M}_n(s) := \sup_W \mathcal{L}_s^n\varphi$. By the last inequality, if $\mathcal{M}_n(s)$ is a function continuous at s_0 for every n, then $\mu_s(\varphi) \to \mu_{s_0}(\varphi)$. Fin an index n. We will prove the continuity of $\mathcal{M}_n(s)$ at s_0.

Let $(s_m) \to s_0$ and $z_m \in W$ so that $\mathcal{L}_{s_m}^n\varphi(z_m) \geq \mathcal{M}_n(s_m) - 1/m$. By extracting a subsequence, we can assume that $z_m \to z_0 \in \overline{W}$. Since $\mathcal{L}_s^n(\varphi)$ is a p.s.h. function of (s, z), by upper semi-continuity property, we have $\limsup \mathcal{L}_{s_m}^n\varphi(z_m) \leq \mathcal{L}_{s_0}^n\varphi(z_0)$. Observe that $\sup_{\overline{W}}\psi = \sup_W \psi$ for all ψ p.s.h. on V since W has smooth boundary. Then $\mathcal{L}_{s_0}^n\varphi(z_0) \leq \mathcal{M}_n(s_0)$. Hence $\limsup_{s\to s_0}\mathcal{M}_n(s) \leq \mathcal{M}_n(s_0)$.

Fixed a positive number ϵ and a no-critical value $x_0 \in W$ of $f_{s_0}^n$ such that $\mathcal{L}_{s_0}^n\varphi(x_0) \geq \mathcal{M}_n(s_0) - \epsilon$. If $r > 0$ is small enough then $B(x_0, 2r)$ is contained in W and does not intersect the set of critical values of f_s^n for s close enough to s_0. We see that $\mathcal{L}_s^n\varphi$ converges to $\mathcal{L}_{s_0}^n\varphi$ in $\mathrm{L}^1(B_r)$, where we denote $B_r := B(x_0, r)$.
Hence when $s \to s_0$, we have

$$\frac{1}{\mathrm{vol}(B_r)}\int_{B_r}\mathcal{L}_s^n\varphi(x)\mathrm{d}x \to \frac{1}{\mathrm{vol}(B_r)}\int_{B_r}\mathcal{L}_{s_0}^n\varphi(x)\mathrm{d}x.$$

On the other hand, the submean inequality gives us

$$\begin{aligned}\sup_{B_r}\mathcal{L}_s^n(\varphi) &\geq \frac{1}{\mathrm{vol}(B_r)}\int_{B_r}\mathcal{L}_s^n\varphi(x)\mathrm{d}x\\ &\geq \frac{1}{\mathrm{vol}(B_r)}\int_{B_r}\mathcal{L}_{s_0}^n\varphi(x)\mathrm{d}x-\epsilon\\ &\geq \mathcal{L}_{s_0}^n\varphi(x_0)-\epsilon\geq\mathcal{M}_n(s_0)-2\epsilon,\end{aligned}$$

for s close enough to s_0. Therefore

$$\liminf_{s\to s_0}\mathcal{M}_n(s)\geq\liminf_{s\to s_0}\sup_{B_r}\mathcal{L}_s^n(\varphi)\geq\mathcal{M}_n(s_0)-2\epsilon.$$

It follows that $\liminf_{s\to s_0}\mathcal{M}_n(s)\geq\mathcal{M}_n(s_0)$. Therefore $\mathcal{M}_n(s)$ is continuous at s_0.

□

It is well-known that the Lyapunov exponent is continuous in the space of rational functions on $\mathbb{C}P^1$ (see [Ma]). We have the following result for families of polynomial-like maps.

Theorem 3.7 *Let $(f_s)_{s\in\Lambda}$ be a holomorphic family of polynomial-like maps as above. If μ_{s_0} is PLB and V is Stein then the sum $L_k(s)$ of all the Lyapunov exponents of f_s is continuous on a neighborhood of s_0.*

Proof. Because μ_s is PLB in a small neighborhood of s_0 then it is sufficient to prove that L_k is continuous at s_0. Define $\varphi_s:=\log|\det\mathrm{Jac}(f_s)|$. Replace V by a Stein open subset of U_{s_0} then we can assume that φ_s is p.s.h. on V. This function is continuous on (s,z) with value in $[-\infty,\infty[$. Since ∂W is smooth, $\sup_W\mathcal{L}_s^n\varphi_s$ is continuous for every n. From inequality (7), we deduce that $L_k(s)=\mu_s(\varphi_s)$ is continuous at s_0.

□

In the case of dimension 1, every family of polynomial-like maps satisfies the hypothesis of Theorem 3.7. We have the following corollary.

Corollary 3.8 *Let $(f_s)_{s\in\Lambda}$ be a holomorphic family of polynomial-like maps as above in dimension one. Then the unique Lyapunov exponent $\chi(s)$ of f_s is continuous.*

We also obtain the following corollary.

Corollary 3.9 *Let $\{f_s : \mathbb{C}P^k \to \mathbb{C}P^k\}_{s\in\Lambda}$ be a holomorphic family of holomorphic endomorphism of $\mathbb{C}P^k$ of algebraic degree $d \geq 2$. Then the sum L_k of all the Lyapunov exponents of f_s is continuous.*

Proof. We can, locally on Λ, lift $(f_s)_{s\in\Lambda}$ to a holomorphic family of homogeneous polynomials $\{F_s : \mathbb{C}^{k+1} \to \mathbb{C}^{k+1}\}_{s\in\Lambda}$. Then by Theorem 3.7, the sum $\tilde{L}_{k+1}(s)$ of all the Lyapunov exponents of F_s is continuous on Λ. Hence, $L_k(s) = \tilde{L}_{k+1}(s) - \log d$ is also continuous.

$\square$

4 Stability of the Julia sets

The purpose of this section is to find some sufficient conditions so that the family of polynomial-like mappings $\{f_s : U_s \to V_s\}_{s\in\Lambda}$ is stable and the sum of all Lyapunov exponents of f_s is a pluriharmonic function. We say that $(f_s)_{s\in\Lambda}$ is stable if the Julia set J_s depends continuously on Λ in the Hausdorff sense.

The stability of the Julia set for rational maps has been studied by Mañé-Sad-Sullivan [MSS] (see also [Mc], [DH], [DM]). Their results can be extended to the case of polynomial-like maps. We have the following result.

Proposition 4.1 *Let $\{f_s : U_s \to V_s\}_{s\in\Lambda}$ be a holomorphic family of polynomial-like maps in dimension one. Let s_0 be a point in Λ. Then the following conditions are equivalent:*

(1) The number of attracting cycles of f_s is locally constant at s_0.

(2) The maximum period of attracting cycles of f_s is locally bounded at s_0.

(3) The Julia set moves holomorphically at s_0.

(4) For s sufficiently close to s_0, every periodic point of f_s is attracting, repelling or persistently indifferent.

(5) The Julia set J_s depends continuously on s (in the Hausdorff topology) on a neighborhood of s_0.

Suppose in addition that there are holomorphic maps $c_i : \Lambda \to \mathbb{C}$ which parameterize the critical points of f_s. The the following condition is also equivalent to those above:

(6) There is a neighborhood U of s_0 such that for s in U, $c_i(s) \in J_s$ if and only if $c_i(s_0) \in J_{s_0}$.

We also have the following theorem in the case of dimension 1.

Theorem 4.2 *Let $\{f_s : U_s \to V_s\}_{s\in\Lambda}$ be a holomorphic family of polynomial-like maps with topological degree $d_t \geq 2$. If $(f_s)_{s\in\Lambda}$ stable in Λ then the unique Lyapunov exponent $\chi(s)$ is a pluriharmonic function.*

Proof. Let $N(s)$ denote the number of critical points of f_s counted without multiplicity. Let $D(f)$ denote the set of $s' \in \Lambda$ such that $N(s)$ does not have a local maximum at s'. This is a proper subvariety of Λ.

If $s_0 \notin D(f)$ then there is a neighborhood Λ_0 of s_0 in Λ and holomorphic functions $c_j : \Lambda_0 \to \mathbb{C}$, $j = 1, 2, \ldots, N$, parameterizing the critical points of f_s (counted with multiplicity).

Therefore, $f'_s(z) = \Pi_{j=1}^{N}(z - c_j(s))h_s(z)$, where $h_s(z)$ is a holomorphic function of (s, z) which does not vanish. Hence,

$$\chi(s) = \sum_{j=1}^{N} \int \log|z - c_j(s)| \mathrm{d}\mu_s + \int \log|h_s(z)| \mathrm{d}\mu_s.$$

By Propositions 2.1 and A.1, $\int \log|h_s(z)|\mathrm{d}\mu_s$ is a pluriharmonic function of s. We want to prove that for each j, the function

$$\lambda_j(s) := \int \log|z - c_j(s)| \mathrm{d}\mu_s$$

is pluriharmonic.

Fix an index j. Using a suitable holomorphic change of coordinate (s, z), we can assume that: $c_j(s) = 0$ for $s \in \Lambda_0$. Then $\lambda_j(s) = \int \log|z|\mathrm{d}\mu_s$. Let $A_{s,n}$ be the set of repelling periodic points $a_{n,i}(s)$ of period n. It follows from Proposition 4.1 that $I_n = \sharp A_{s,n}$ is independent of s and $a_{n,i}(s)$ is a holomorphic function of s, for $1 \leq i \leq I_n$. Define

$$\mu_{s,n} := \frac{1}{d_t^n} \sum_{i=1}^{I_n} \delta_{a_{n,i}(s)}.$$

By Proposition 4.1, either $0 \in J_s$ for every $s \in \Lambda_0$ or $0 \notin J_s$ for every $s \in \Lambda_0$. Let $\overline{\triangle(r)}$ denote the closed disk of center 0 and of radius r with r small. We

define for each positive fixed number r,

$$\begin{aligned} I_{n,r} &:= \left\{i : a_{n,i}(s_0) \notin \overline{\triangle(r)}\right\} \\ \mu_{s,n,r} &:= \frac{1}{d_t^n} \sum_{i \in I_{n,r}} \delta_{a_{n,i}(s)} \\ \mu_{s_0,r} &:= \mu_{s_0}\big|_{J_{s_0} \setminus \overline{\triangle(r)}} \\ \lambda_{j,r}(s) &:= \limsup_{n\to\infty} \int \log|z| \mathrm{d}\mu_{s,n,r}. \end{aligned}$$

By Proposition 4.1, there exist holomorphic motions $\{\phi_s : J_{s_0} \to \mathbb{C}\}_{s\in\Lambda_0}$ such that: $\phi_s \circ f_{s_0}(z) = f_s \circ \phi_s(z)$, $\phi_s(J_{s_0}) = J_s$, $\phi_s(a_{n,i}(s_0)) = a_{n,i}(s)$ and if $0 \in J_s$ then $\phi_s(0) = 0$. We have

$$a_{n,i}(s) \in J_s \setminus \phi_s(J_{s_0} \cap \overline{\triangle(r)}) \quad \text{for all } i \in I_{n,r}.$$

It follows that $\int \log|z|\mathrm{d}\mu_{s,n,r}$ is pluriharmonic on $s \in \Lambda_0$ and bounded by a constant independent of n. Put $\mu_{s,r} := \mu_s|_{J_s\setminus\phi_s(J_{s_0}\cap\overline{\triangle(r)})}$. We have $\mu_{s,n,r} \rightharpoonup \mu_{s,r}$ as $n \to \infty$. These measures have support out of a neighborhood of 0. Hence $\int \log|z|\mathrm{d}\mu_{s,n,r}$ converges to $\int \log|z|\mathrm{d}\mu_{s,r}$. This implies that $\lambda_{j,r}(s)$ is pluriharmonic on $s \in \Lambda_0$. When $r > 0$ is small and decreases to 0, $\lambda_{j,r}(s)$ decreases to $\lambda_j(s)$. Hence $\lambda_j(s)$ is pluriharmonic on Λ_0. Therefore, $\chi(s)$ is a pluriharmonic function on $\Lambda\setminus D(f)$. By Corollary 3.8, $\chi(s)$ is continuous on Λ. It implies that $\chi(s)$ is pluriharmonic on Λ.

□

The following result is valid in any dimension.

Theorem 4.3 *Let $\{f_s : U_s \to V_s\}_{s\in\Lambda}$ be a holomorphic family of polynomial-like maps of topological degree $d_t \geq 2$ and $\mathcal{C}_s$ the critical set of f_s. Assume that μ_s is PLB and $\mathcal{C}_s \cap J_s = \varnothing$ for $s \in \Lambda$. Then*

(i) The sum $L_k(s)$ of all the Lyapunov exponents of f_s is a pluriharmonic function. In particular, the bifurcation locus is empty.

(ii) The family $(f_s)_{s\in\Lambda}$ is stable.

Note that a polynomial-like mapping satisfying the condition $\mathcal{C}_s \cap J_s = \varnothing$ is, in general, not uniformly hyperbolic on J_s.

By [DS1], f_s admits repelling periodic points on J_s. We have the following lemma, see [FS1] for the proof.

Lemma 4.4 *Let $\{f_s : U_s \to V_s\}_{s\in\Lambda}$ be a holomorphic family of polynomial-like maps of topological degree $d_t \geq 2$. Then for all $s_0 \in \Lambda$, there exists a neighborhood Λ_{s_0} of s_0, a positive integer N and repelling periodic points $p(s) \in J_s$ such that $p(s)$ depends holomorphically on $s \in \Lambda_{s_0}$ and $f_s^N(p(s)) = p(s)$.*

Proof of Theorem 4.3. (i) Let $\mathcal{E}_s$ denote the exceptional set of f_s, i.e the set of point $z \in V_s$ such that the measure $d_t^{-n} \sum_{f_s^n(w)=z} \delta_w$ does not converge to μ_s. Since μ_s is PLB then $\mathcal{E}_s$ is contained in the postcritical set $\bigcup_{n\geq 1} f_s^n(C_s)$ of f_s for all $s \in \Lambda$ (see [DS1]). Hence $\mathcal{E}_s \cap J_s = \varnothing$. Fix a point s_0 in Λ and let $p(s)$ be as in Lemma 4.4. Define

$$\mu_{s,n} = \frac{1}{d_t^n} \sum_{i=1}^{d_t^n} \delta_{p_{n,i}(s)}$$

where $p_{n,i}(s)$ are preimages of $p(s)$ by f_s^n. Since J_s is invariant by f_s^{-1}, all the points $p_{n,i}(s)$ are in J_s. The condition $\mathcal{C}_s \cap J_s = \varnothing$ implies that $p_{n,i}(s)$ depends holomorphically on Λ_{s_0} and $\log|\det \mathrm{Jac}(f_s)|$ is pluriharmonic on a neighborhood of J_s. These and the property that $\mu_{s,n} \rightharpoonup \mu_s$ imply

$$\begin{aligned} L_k(s) &= \int \log|\det \mathrm{Jac}(f_s)| \mathrm{d}\mu_s \\ &= \lim_{n\to\infty} \int \log|\det \mathrm{Jac}(f_s)| \mathrm{d}\mu_{s,n} \\ &= \frac{1}{d_t^n} \lim_{n\to\infty} \sum_{i=1}^{d_t^n} \log|\det \mathrm{Jac}(f_s)(p_{n,i}(s))|. \end{aligned}$$

Therefore $L_k(s)$ is pluriharmonic.

(ii) Observe that the family of holomorphic maps $p_{n,i} : \Lambda_{s_0} \to W$ is normal where W is an open set such that $J_s \subset W \Subset \mathbb{C}^k$ for $s \in \Lambda_{s_0}$. Consider the family $\mathcal{F}$ of all the maps $v : \Lambda_{s_0} \to W$ that we obtain as limit, locally uniformly on Λ_{s_0}, of a subsequence of $(p_{n,i})$. Hence $\mathcal{F}$ is a normal family and $\bigcup_{v\in\mathcal{F}} v(s) = J_s$ since $\mu_{s,n} \rightharpoonup \mu_s$. It follows that $(f_s)_{s\in\Lambda}$ is stable. □

Corollary 4.5 *Let $\{f_s : \mathbb{C}P^k \to \mathbb{C}P^k\}_{s\in\Lambda}$ be a holomorphic family of holomorphic endomorphisms of algebraic degree $d \geq 2$. Assume that $\mathcal{C}_s \cap J_s = \varnothing$ for $s \in \Lambda$, where $\mathcal{C}_s$ is the critical set of f_s. Then*

(i) The sum $L_k(s)$ of all the Lyapunov exponents of f_s is pluriharmonic on Λ.

(ii) The family $(f_s)_{s\in\Lambda}$ is stable.

Proof. We can, locally on Λ, lift $(f_s)_{s\in\Lambda}$ to a holomorphic family of homogeneous polynomials $\{F_s : \mathbb{C}^{k+1} \to \mathbb{C}^{k+1}\}_{s\in\Lambda}$. Since $\mathcal{C}_s \cap J_s = \varnothing$, it implies that $\tilde{\mathcal{C}}_s \cap \tilde{J}_s = \varnothing$ where $\tilde{\mathcal{C}}_s$ and $\tilde{J}_s$ are the critical set and the Julia set of F_s respectively. By Theorem 4.3, the sum $\tilde{L}_{k+1}(s)$ of all the Lyapunov exponents of F_s is pluriharmonic on Λ and $(F_s)_{s\in\Lambda}$ is stable. Hence, $L_k(s) = \tilde{L}_{k+1}(s) - \log d$ is also pluriharmonic.

If $\pi : \mathbb{C}^{k+1}\backslash\{0\} \to \mathbb{C}P^k$ denotes the canonical projection, we have $J_s = \pi(\tilde{J}_s)$. Note that there is a neighborhood of 0 in $\mathbb{C}^{k+1}$ which does not intersect $\tilde{J}_s$. Since $(F_s)_{s\in\Lambda}$ is stable then $(f_s)_{s\in\Lambda}$ is also stable.

□

A Appendix: horizontal currents

Let Λ and V be two bounded open subsets of $\mathbb{C}^m$ and $\mathbb{C}^k$ respectively. Let π and π_V denote the canonical projections of $\Lambda \times V$ on Λ and on V. Let R be a positive closed current of bidegree (k,k) on $\Lambda \times V$. We say that R is *horizontal* if $\pi_V(\mathrm{supp}(R)) \Subset V$. Dinh-Sibony (see [DS2]) proved that the slice measure $\langle R, \pi, s\rangle$ is defined for every s and its mass is independent of s. We call this mass the *slice mass* of R. We can consider $\langle R, \pi, s\rangle$ as the intersection (wedge-product) of the current R with the current $[\pi^{-1}(s)]$ of integration on $\pi^{-1}(s)$. The slice measure are characterized by the following formula:

$$\int_\Lambda \langle R, \pi, s\rangle(\psi)\Omega(s) = \langle R \wedge \pi^*(\Omega), \psi\rangle, \tag{8}$$

for every continuous (m,m)-form Ω with compact support in Λ and every continuous test function ψ on $\Lambda \times V$. Note that the formula (8) is valid in the general case where π is an holomorphic submersion between two complex manifolds $\mathcal{V}$ and Λ.

More over, we have the following proposition.

Proposition A.1 *Let R be a horizontal positive closed current on $\Lambda \times V$ and ψ be a p.s.h. function on a neighborhood of* $\operatorname{supp}(R)$. *Then the function*

$$\varphi(s) := \int \psi(s, \cdot)\langle R, \pi, s\rangle$$

is p.s.h. on Λ or equal to $-\infty$ identically.

Proof. This Proposition is a consequence of [DS2, Theorem 2.1], where the authors consider the case of continuous p.s.h. function ψ. We obtain the general case by using a sequence of smooth p.s.h. functions decreasing to ψ. □

We refer to [De], [FS2] and [DS2] for the theory of intersection of currents. We now prove the following theorem.

Theorem A.2 *Let R be a horizontal positive closed current. Let u be a p.s.h. function on $\Lambda \times V$. Assume there exists $s_0 \in \Lambda$ such that $\langle R, \pi, s_0\rangle(u) \neq -\infty$. Then the current uR has locally finite mass in $\Lambda \times V$. In particular, the positive closed current* $dd^c u \wedge R$ *is well defined.*

Proof. Consider open sets $\Lambda_0 \Subset \Lambda$, $V_0 \Subset V$ such that $s_0 \in \Lambda_0$ and R is a horizontal positive closed current on $\Lambda_0 \times V_0$. Since the problem is local then it is sufficient to prove that uR has locally finite mass in $\Lambda_0 \times V_0$.

Let A be a matrix of size $m \times k$ with complex coefficients. We can consider A as a point in $\mathbb{C}^{mk}$. Define the affine map $H : \mathbb{C}^{mk} \times \mathbb{C}^m \times \mathbb{C}^k \to \mathbb{C}^m \times \mathbb{C}^k$ by

$$H(A, s, z) := (s - Az, z).$$

There exists a small ball $B(0, r)$ in $\mathbb{C}^{mk}$ such that $\mathcal{R} := H^*(R)$ is a horizontal positive closed current on $\tilde{\Lambda} \times V_0$, where $\tilde{\Lambda} := B(0, r) \times \Lambda_0$.

Define $\tilde{u} : \tilde{\Lambda} \times V_0 \to \mathbb{R} \cup \{-\infty\}$ by $\tilde{u}(A, s, z) := u(s + Az, z)$. Then $\tilde{u}$ is p.s.h. on $\tilde{\Lambda} \times V_0$. Consider $v : B(0, r) \times V_0 \to \mathbb{R} \cup \{-\infty\}$ by $v(A, s) := \langle \mathcal{R}, \pi_{\tilde{\Lambda}}, (A, s)\rangle(\tilde{u})$, where $\pi_{\tilde{\Lambda}}$ denotes the canonical projection of $\tilde{\Lambda}_0 \times V_0$ on $\tilde{\Lambda}$. By Proposition A.1, $v(A, s)$ is a p.s.h. function or is equal $-\infty$ identically. But $v(0, s_0) = \langle \mathcal{R}, \pi_{\tilde{\Lambda}}, (0, s_0)\rangle(\tilde{u}) = \langle R, \pi, s_0\rangle(u) \neq -\infty$ then v is a p.s.h. function.

Define $\pi_A : \Lambda_0 \times V_0 \to V$ by $\pi_A(s, z) := s + Az$. Then π_A is a linear projection of $\Lambda_0 \times V_0$ on Λ. For $s \in \Lambda_0$, we have

$$\langle R, \pi_A, s\rangle(u) = v(A, s).$$

Then $s \mapsto \langle R, \pi_A, s\rangle(u)$ is a p.s.h. function of $s \in \Lambda_0$ for all $A \in B(0,r)\backslash\mathcal{E}$, where $\mathcal{E} \subset B(0,r)$ is a pluripolar set. This implies that $s \mapsto \langle R, \pi_A, s\rangle(u)$ is locally integrable in Λ_0. Let K be a compact subset of $\Lambda_0 \times V_0$. Then by the formula (8), for $A \in B(0,r)\backslash\mathcal{E}$, we have

$$\|uR \wedge \pi_A^*(i\mathrm{d}s_1 \wedge \mathrm{d}\bar{s}_1 \wedge \ldots \wedge i\mathrm{d}s_m \wedge \mathrm{d}\bar{s}_m)\|_K < \infty. \tag{9}$$

We can obtain a strictly positive form on K by taking a combination of the forms $\pi_A^*(i\mathrm{d}s_1 \wedge \mathrm{d}\bar{s}_1 \wedge \ldots \wedge i\mathrm{d}s_m \wedge \mathrm{d}\bar{s}_m)$ for $A \in B(0,r)\backslash\mathcal{E}$. By the inequality (9), hence uR has locally finite mass in $\Lambda_0 \times V_0$. This implies the theorem. We define $\mathrm{dd^c}u \wedge R := \mathrm{dd^c}(uR)$, see e.g [FS2].

□

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Ngoc-Mai Pham
Mathématique- Bâtiment 425, UFR 8628
Université Paris-Sud, 91405 Orsay, France
`Ngocmai.Pham@math.u-psud.fr`